Computational Fluid Dynamics JOURNAL vol.2 no.4 January 1994 (375 — 392)

DISCRETE APPROXIMATION OF SINGULARLY PERTURBED PARABOLIC PDES WITH A DISCONTINUOUS INITIAL CONDITION

P.W. HEMKER[†] G.I. SHISHKIN[‡]

Abstract

In this paper a Dirichlet problem for a parabolic partial differential equation with a discontinuous initial condition is studied. The second order derivative is multiplied by a small parameter ε, with an arbitrary value in $(0, 1]$. When this parameter vanishes, the PDE degenerates to a first order equation with respect to the time variable t. We study the case where the boundary condition for $t = 0$ has a discontinuity of the first kind, so that an internal layer appears. On a uniform rectangular grid we construct a fitted difference scheme that converges in the discrete ℓ^∞-norm over the whole domain, uniformly in the small parameter.

For a fitted difference schemes we compare the theoretical results and the results in practice, and we show the essential differences with those obtained for the classical scheme. The order of convergence in the numerical experiment is determined and we observe that the new, adapted (fitted) approximation converges on the whole domain of definition, uniformly in ε. We show that this is in contrast to the approximation obtained by the classical scheme, which for a fixed ε and on a uniform grid does not converge in the discrete ℓ^∞-norm in a neighbourhood of the discontinuous boundary condition and it also does not converge uniformly in ε in the interior layer.

1. INTRODUCTION

The solution of partial differential equations that are singularly perturbed and / or have discontinuous boundary conditions generally have only limited smoothness. Due to this fact difficulties appear when we solve these problems by numerical methods. For example for regular parabolic equations with discontinuous boundary conditions, classical methods (FDM or FEM) on regular rectangular grids do not converge in the ℓ^∞-norm on a domain that includes a neighbourhood of the discontinuity [1, 2, 3].

If the parameter, ε, multiplying the highest-order derivative vanishes, boundary- and interior layers generally appear. When a discontinuity is present in the initial function (given at $t = 0$), an interior layer is generated. Outside a neighbourhood of the discontinuity classical difference schemes converge in the ℓ^∞-norm for each fixed value of the small parameter, but they do not converge in the ℓ^∞-norm in the neighbourhood of the discontinuity. Neither do they converge uniformly in ε in any neighbourhood of the interior layer [1, 2]. Therefore, it is interesting

[†] Centrum voor Wiskunde en Informatica, Amsterdam, The Netherlands

[‡] Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Science, Ekaterinburg, Russia

to construct special methods which are ℓ^∞-convergent for parabolic PDEs with discontinuous initial functions, both in the regular and in the singularly perturbed case. In the latter case it is important to see if and when such convergence can be uniform for the small parameter on the whole domain of definition.

To construct numerical methods that handle problems with singularities, generally one of the following three approaches [4] can be used: 1. the singularity is split off and represented separately by a special numerical technique (e.g. in the Finite Element Method by introducing special basis functions); 2. the mesh is properly refined in the neighbourhood of the singularity; or 3. the coefficients of the difference equations are fitted to the singularity.

In [1, 2] singularly perturbed parabolic equations with discontinuous boundary conditions were studied. There, special difference schemes were constructed for these problems. In order to be able to construct a method that was uniformly convergent (in the small parameter ε), special variables were used in the neighbourhood of the discontinuity. By introducing the variables $\theta = x/(2\varepsilon\sqrt{t})$ and t, the singularity was removed from the boundary value problem and the solution became a smooth function in the new variables. This behaviour of the transformed solution allows the use of a classical scheme in the transformed variables in the neighbourhood of the singularity. Away from the singularity the classical scheme can be used with the original variables.

This transformation in the neighbourhood of the singularity implied the use of a specially condensed grid in the neighbourhood of the boundary and interior layers. So we can say that the technique used in [1, 2] combines the approaches mentioned under 2 and 3. For these schemes ℓ^∞-convergence on the whole domain is proved, uniformly in the small parameter, but a disadvantage of these schemes is that they are very hard to realise in practice.

Because fitting of the coefficients, combined with fitting of the mesh is generally too complex for practical application, in the present paper we propose a new method in which only the coefficients are adapted. We use a uniform rectangular grid and a special difference equation with fitted coefficients. This method is much easier to realise.

For the construction of the new scheme the coefficients are selected such that the solution of a model problem with a piecewise constant, discontinuous initial function is the exact solution of the difference equations.

This difference scheme with adapted coefficients is studied in this paper and it is compared with the classical scheme.

As was shown in [1, 2], no scheme exists that converges uniformly on a uniform grid for the general problem with a parabolic layer. However, for problems with an interior layer, that originates at a discontinuous boundary condition, the present method has this favourable property, and –moreover– numerical examples show that the method has practical value for far more general equations with discontinuous boundary conditions.

2. PROBLEM FORMULATION

We consider the Dirichlet boundary value problem for the following singularly perturbed equation of parabolic type[1]

$$\begin{aligned} L_{(1)}u(x,t) = f(x,t),\ (x,t)\in G, \\ u(x,t) = \phi(x,t),\ (x,t)\in S, \end{aligned} \tag{1a}$$

where

$$G = \{(x,t) \mid -1 < x < 1,\ 0 < t \le T\}, \quad S = \overline{G} \backslash G, \tag{1b}$$

$$L_{(1)} \equiv \varepsilon^2 \frac{\partial^2}{\partial x^2} - p(t)\frac{\partial}{\partial t} - c(t)\,. \tag{1c}$$

The parameter ε may take any value $\varepsilon\in(0,1)$. The coefficients $c(t)$, $p(t)$ and the source $f(x,t)$ are sufficiently smooth functions on $\overline{G}$ and the coefficients are positive:

$$c(t) \ge 0, \quad p(t) \ge p_0 > 0, \quad (x,t)\in\overline{G}. \tag{2}$$

The boundary function $\phi(x,t)$ has a discontinuity[2] of the first kind on the set $S^\star$:

$$S^\star = \{(x,t) \mid x = 0, t = 0\}.$$

For simplicity $S^\star$ consists of a single point only. Outside $S^\star$ the function $\phi(x,t)$ is sufficiently smooth on S.

Such boundary value problems with discontinuous boundary condition describe e.g. the temperature in a heat transfer problem, when two parts of a material with different temperatures are instantaneously connected [5]. Then, the small parameter ε corresponds with a small heat conduction coefficient.

The solution of the boundary value problem (1) is a function $u\in C(\overline{G}\backslash S^\star) \cap C^{2,1}(G)$, i.e. on G it is C^2 in x and C^1 in t.

We say that the discrete approximation converges ε-uniformly (or uniformly in ε) on $\overline{G}$ if the ℓ^∞-norm of the error converges to zero on $\overline{G}$, uniformly in ε.

3. THE BEHAVIOUR OF THE SOLUTION AND ITS DERIVATIVES

In order to see what are the difficulties with the classical difference schemes, to be able to construct the special difference scheme for our problem (1) and in order to study its behaviour,

[1]The subscript number (within brackets) for a symbol denotes the equation in which the symbol is defined.

[2]A piecewise continuous function $v(x,t)$, $(x,t)\in S\backslash S^\star$, is redefined at the discontinuity by

$$v(x,t) = \frac{1}{2}\left\{\lim_{s\searrow 0} v(x+s,t) + \lim_{s\nearrow 0} v(x+s,t)\right\}, (x,t)\in S^\star, \tag{3}$$

and the jump in the discontinuity is defined by

$$[v(x,t)] = \left\{\lim_{s\searrow 0} v(x+s,t) - \lim_{s\nearrow 0} v(x+s,t)\right\}, (x,t)\in S^\star. \tag{4}$$

we first need some estimates for the solution and its derivatives.

To start with, let us take the parameter ε fixed. Then we notice that, the solution of problem (1) is continuous on $\overline{G}\backslash S^*$. The discontinuity appears only at the point $(0,0)$. The derivatives exist and are bounded in $\overline{G}$, outside a neighbourhood of S^*. They only increase, without bound, in the vicinity of S^*. When the parameter tends to zero, an interior layer appears and the derivatives with respect to x increase without bound, also inside G in the neighbourhood of the interior layer.

Due to the maximum principle [6] we have for the solution of (1) the estimate

$$|u(x,t)| \leq M, \quad (x,t) \in G, \tag{5}$$

where

$$M = (p_0)^{-1} T \max_{\overline{G}} |f(x,t)| \; + \; \max_{S} |\phi(x,t)| \, .$$

Here and in the following, by M (or m) we denote a sufficiently large (small) positive constant not depending on the small parameter ε. In the case of difference problems these constants do not depend on the parameters of the grid either. The constants do not necessarily represent the same value at different appearances.

We introduce the standard function $w_0(x,t)$, which is discontinuous in S^*,

$$w_0(x,t) = w_0(x,t;p_1) = \frac{1}{2} v\Big(\frac{x}{2\varepsilon} \sqrt{\frac{p_1}{t}} \Big), \quad (x,t) \in \overline{G}\backslash S^* , \tag{6}$$

where $p_1 = p(0)$ and $v(\xi) = \operatorname{erf}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^\xi \exp(-\alpha^2) \, d\alpha$ is the error function. For $t = 0$, in point $x = 0$, the function (6) is defined by continuous extension. The function $w_0(x,t)$ is the solution of the constant coefficient equation

$$L_{(7)} \, u(x,t) \equiv \left(\varepsilon^2 \frac{\partial^2}{\partial x^2} - p_1 \frac{\partial}{\partial t} \right) \, u(x,t) = 0, \quad (x,t) \in G. \tag{7}$$

This function is piecewise constant on S at $t = 0$ and has a discontinuity of the first kind in S^* :

$$[w_0(0,0)] = 1.$$

Suppose

$$W(x,t) = [\phi(0,0)] \; W_0(x,t), \quad (x,t) \in \overline{G}\backslash S^*, \tag{8}$$

where

$$W_0(x,t) = \exp\left(- \int_0^t \frac{c(\xi)}{p(\xi)} d\xi \right) w_0(x, \eta(t); p_1), \quad \text{with } \eta(t) = \int_0^t \frac{p_1}{p(\xi)} \, d\xi \, . \tag{9}$$

Then the function $W(x,t)$ is continuous on $\overline{G}\backslash S^*$, it is a solution of the homogeneous equation

$$L_{(1)} \, u(x,t) = 0, \quad (x,t) \in G \, , \tag{10}$$

and it has a jump at $S^\star$:

$$[W(x,t)] = [u(x,t)] = [\phi(x,t)], \quad (x,t)\in S^\star.$$

We now write the solution of problem (1) as a sum

$$u(x,t) = W(x,t) + U(x,t), \quad (x,t)\in\overline{G}\backslash S^\star. \tag{11}$$

Here, $W(x,t)$ is the solution of the problem

$$\begin{array}{c} L_{(1)}\, u(x,t) = 0, \quad (x,t)\in G, \\ u(x,t) = W(x,t),\ (x,t)\in S, \end{array} \tag{12}$$

and $U(x,t)$ is the solution of the problem

$$\begin{array}{c} L_{(1)}\, u(x,t) = f(x,t),\ (x,t)\in G, \\ u(x,t) = \phi(x,t) - W(x,t),\ (x,t)\in S. \end{array} \tag{13}$$

On S the function $U(x,t)$ is continuous and piecewise smooth. For simplicity we suppose that $U(x,t)$ is sufficiently smooth on the boundary of G, and that a compatibility condition is satisfied at the corner points. We are interested in the solution of problem (1) in the neighbourhood of the point of discontinuity and in the neighbourhood of the generated interior layer. Therefore, we suppose that the boundary conditions at $x = \pm 1$ are such that no boundary layers appear. Then we have the estimates

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} U(x,t)\right| \le M\left\{1 + \varepsilon^{1-k} t^{1/2-(k_0+k/2)} \exp\left(-\frac{x}{2\varepsilon}\sqrt{\frac{p}{t}}\right)\right\}, \quad (x,t)\in\overline{G}, \tag{14}$$

and for $W(x,t)$, the singular part of the solution, we derive for $k, k_0 \ge 0$,

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} W(x,t)\right| \le M\varepsilon^{-k}\, t^{-(k_0+k/2)} \exp\left(-\frac{x}{2\varepsilon}\sqrt{\frac{p}{t}}\right), \ (x,t)\in\overline{G}\backslash S^\star. \tag{15}$$

These bounds are determined by means of a priori estimates, see e.g. [7, 8] for the regular and [1, 2] for the singularly perturbed problems. Thus, for the singular and the regular parts of the solution we have the estimates (14) and (15), respectively.

Notice that, with

$$0 < \alpha_0 \le \left|\frac{x^2}{\varepsilon^2 t}\right| \le \alpha_1 < \infty, \quad k + 2k_0 \ge 1,$$

we have

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} W(x,t)\right| \ge m\varepsilon^{-k} t^{-(k_0+k/2)} \to \infty, \quad \text{for } x,t \to 0, \tag{16}$$

for any value of the parameter ε, and with

$$\left|\frac{x^2}{\varepsilon^2}\right| \le \alpha, \quad t \ge t_0 > 0,$$

we have

$$\left|\frac{\partial^k}{\partial x^k}W(x,t)\right| \geq m\varepsilon^{-k} \longrightarrow \infty\,, \quad \text{for } \varepsilon \longrightarrow 0 \tag{17}$$

for any fixed $\alpha > 0$. This means that in the neighbourhood of the set $S^\star$ the derivatives of the solution are unbounded, for any fixed value of ε and also that the space derivatives are unbounded, uniformly in ε, in the neighbourhood of the interior layer.

4. AN ε-UNIFORMLY CONVERGENT SCHEME

On the set $\overline{G}$ we introduce the rectangular grid

$$\overline{G}_h = \omega \times \omega_0\,. \tag{18}$$

Here ω and ω_0 are uniform grids on the segments $[-1,1]$ and $[0,T]$ respectively. For some $N, N_0 > 0$ we take $x_i = ih$, $i\in\mathbb{Z}$; $-1 \leq x_i \leq 1$; $h = 2/N$; $t^j = j\tau$; $j = 0,1,2,\cdots,N_0$, $\tau = T/N_0$; and

$$G_h = G \cap \overline{G}_h; \quad S_h = S \cap \overline{G}_h; \quad S_h^\star = S^\star \cap \overline{G}_h.$$

On the set $S_h^\star$ the boundary function $\phi(x,t)$ is defined by

$$\phi(x,t) = \frac{1}{2}\left\{\lim_{s\nearrow x}\phi(s,t) + \lim_{s\searrow x}\phi(s,t)\right\}, \quad (x,t)\in S_h^\star. \tag{19}$$

For the numerical approximation of (1) we may use the classical difference approximations (see e.g. [9, 10]). E.g. in the case of the implicit central difference scheme we have

$$\begin{aligned} \Lambda_{(20)}\, z(x,t) &= f(x,t),\ (x,t)\in G_h,\\ z(x,t) &= \phi(x,t),\ (x,t)\in S_h, \end{aligned} \tag{20a}$$

where

$$\Lambda_{(20)} \equiv \varepsilon^2\delta_{x\bar{x}} - p(t)\delta_{\bar{t}} - c(t), \tag{20b}$$

with $\delta_{\bar{t}}z(x,t)$ and $\delta_{x\bar{x}}z(x,t)$ the usual first and second difference of $z(x,t)$ on the uniform grids ω_0 and ω respectively; the bar denotes the backward difference. It is well known that the operator $\Lambda_{(20)}$ is monotone [10]. It implies that the maximum principle holds for (20).

Nevertheless, the classical difference scheme (i) does not converge on the whole domain $\overline{G}_h^\star = \overline{G}_h \backslash S_h^\star$ for a fixed value of ε, and (ii) outside a neighbourhood of the discontinuity it does not converge uniformly with respect to ε in the interior layer (see Section 5). To obtain uniform convergence, in the present paper we introduce a specially fitted scheme for the approximation of equation (1a),

$$\begin{aligned} \Lambda_{(21)} z(x,t) &= f(x,t),\ (x,t)\in G_h,\\ z(x,t) &= \phi(x,t),\ (x,t)\in S_h, \end{aligned} \tag{21a}$$

where

$$\Lambda_{(21)} \equiv \varepsilon^2\gamma(x,t)\delta_{x\bar{x}} - p(t)\delta_{\bar{t}} - c(t)\,. \tag{21b}$$

According to the principle mentioned in the introduction, here $\gamma(x,t)$ is a fitting coefficient, which is chosen such that the singular solution, $W(x,t)$, is the exact solution of the homogeneous difference equation (22):

$$\Lambda_{(21)}W(x,t) \equiv \left\{\varepsilon^2\gamma(x,t)\delta_{x\bar{x}} - p(t)\delta_{\bar{t}} - c(t)\right\} W(x,t) = 0, \quad (x,t)\in G_h\,. \tag{22}$$

More generally we can select $\gamma(x,t)$ such that (22) is satisfied by $v(x,t) = W(x,t)+u_0(x,t)$, where W is the principal part of the singular solution and u_0 is some smooth solution of the homogeneous equation

$$L_{(1)}\, u(x,t) = 0, \quad (x,y)\in G\,. \tag{23}$$

This leads to the following expression for γ:

$$\gamma(x,t) = \frac{p(t)\delta_{\bar{t}}v(x,t) + c(t)v(x,t)}{\varepsilon^2\delta_{x\bar{x}}v(x,t)}, \quad (x,t)\in G_h\,, \tag{24}$$

for any point (x,t) where $\delta_{x\bar{x}}v(x,t) \neq 0$. Because of the linearity of (23) we can use

$$v(x,t) = W_0(x,t) + u_0(x,t), \quad (x,t)\in G_h. \tag{25}$$

We notice that for $u_0(x,t) \equiv 0$ the differences $\delta_{x\bar{x}}v(x,t)$ and $\delta_{\bar{t}}v(x,t)$ can be very small because of the exponentially small derivatives of $W_0(x,t)$ for large $x/(\varepsilon\sqrt{t})$. To improve the numerical behaviour in the computation of $\gamma(x,t)$, we choose the function u_0 such that the differences $\delta_{x\bar{x}}W_0$ and $\delta_{x\bar{x}}u_0$ have the same sign, for $(x,t)\in G_h$. In particular, in the remaining part of this paper we take e.g.

$$u_0(x,t) = -\left\{x^3 + 6\varepsilon^2 x \int_0^t \frac{1}{p(\xi)}\,d\xi\right\} \exp\left(-\int_0^t \frac{c(\nu)}{p(\nu)}d\nu\right), \quad (x,t)\in\overline{G}, \tag{26}$$

so that, for example for $c(t) \equiv 0$ and $p(t) \equiv 1$, we obtain

$$u_0(x,t) = u_{(27)}(x,t) = -x^3 - 6\varepsilon^2 xt\,, \quad (x,t)\in\overline{G}\,. \tag{27}$$

Then, for $\gamma(x,t)$ we have the general representation

$$\gamma(x,t) = \frac{p(t)(\delta_{\bar{t}}W_0(x,t) + \delta_{\bar{t}}u_0(x,t)) + c(t)(W_0(x,t) + u_0(x,t))}{\varepsilon^2\delta_{x\bar{x}}W_0(x,t) + \varepsilon^2\delta_{x\bar{x}}u_0(x,t)}, x \neq 0\,, \tag{28}$$

where the functions W_0 and u_0 are defined by (9) and (26) respectively. We notice that $\delta_{x\bar{x}}v = \delta_{\bar{t}}v = v = 0$ at $x = 0$, $t > 0$. Therefore, for definiteness we set $\gamma(x,t) = 1$ for $x = 0$. [3] Now we define the resulting difference scheme as (21), where $\gamma(x,t)$ is determined by (28).

It requires a long and tedious computation to derive the error estimate for the scheme (21), (28). It is derived along the lines as given in [1, 2, 3] from expression (11) along the following lines. Using (28) we find estimates for $|\gamma(x,t)|$ and $|\gamma(x,t)-1|$. Then, applying the maximum

[3] According to (9) for computation $\delta_{x\bar{x}}W_0(x,t)$ on time layer $t = j\tau$ we use difference derivatives $\delta_{x\bar{x}}w_0(x,\eta(t))$. The difference derivative $\delta_{\bar{t}}W_0(x,t)$, $\delta_{x\bar{x}}W_0(x,t)$, $\delta_{\bar{t}}u_0(x,t)$, $\delta_{x\bar{x}}u_0(x,t)$ can easily be found e.g. when function $p(t)$, $c(t)$ are analytical.

principle for the difference problem, we estimate the error for the function

$$|e_{(29)}(x,t)| = z_U(x,t) - U(x,t)\,, \quad (x,t)\in\overline{G}_h\,, \tag{29}$$

where $U(x,t)$ is the smooth part in (11) and $z_U(x,t)$ is its representation obtained by the difference approximation. We obtain the estimates

$$|e_{(29)}(x,t)| \le M\left\{t^{\frac{1}{2}} + \frac{\tau^{3/2}}{h}\right\}, \quad (x,t)\in\overline{G}_h\,, \tag{30}$$

and

$$|e_{(29)}(x,t)| \le M\left\{\rho^{\frac{1}{2}} + \frac{\tau^{3/2}}{h}\right\}, \quad (x,t)\in\overline{G}_h\,, \quad t\le\rho\,. \tag{31}$$

Then we estimate $e_{(29)}(x,t)$ for $t \ge \rho$. We apply the maximum principle [10] for $t \ge \rho$, take into account the estimate (14) and the truncation error

$$|\Lambda_{(21)}e_{(29)}(x,t)| \le M\rho^{-2}(h+\tau)\,, \quad (x,t)\in\overline{G}_h\,, \tag{32}$$

if $t \ge \rho$. So, we get

$$|e_{(29)}(x,t)| \le M\left\{\rho^{\frac{1}{2}} + \rho^{-\alpha}(h+\tau) + \frac{\tau^{3/2}}{h}\right\}, \quad (x,t)\in\overline{G}_h\,, \quad t\ge\rho\,. \tag{33}$$

for any $\alpha\in(1,2)$. Because ρ is arbitrary in this inequality, we arrive at the estimate

$$|z_U(x,t) - U(x,t)| \le M\left\{(h+\tau)^{\nu} + \frac{\tau^{3/2}}{h}\right\}, \quad (x,t)\in\overline{G}_h\,, \tag{34}$$

for any $\nu\in(0,1/3)$. In a similar way we find a similar estimate for the error

$$|e_{(35)}(x,t)| = z_W(x,t) - W(x,t)\,, \quad (x,t)\in\overline{G}_h\,, \tag{35}$$

where $W(x,t)$ is the singular part of (11) and $z_W(x,t)$ is its representation obtained by the difference approximation. Both, estimates lead to the following error estimate for problem (21), (28).

$$\max_{\overline{G}_h}|u(x,t) - z(x,t)| \le M\left\{(h+\tau)^{\nu} + \frac{\tau^{3/2}}{h}\right\}, \tag{36}$$

for any $\nu\in(0,1/3)$. Let h and τ decrease such that

$$\frac{\tau^{3/2}}{h} \le \psi(h,\tau) \tag{37}$$

where $\psi(h,\tau) > 0$ and $\psi(h,t) \to 0$ for $h,\tau\to 0$, then the scheme (21), (28) converges uniformly in ε:

$$\max_{\overline{G}_h}|u(x,t) - z(x,\tau)| \le M\left\{(h+\tau)^{\nu} + \psi(h,\tau)\right\}. \tag{38}$$

If, for instance,

$$h \geq \mathcal{O}(\tau^{\frac{3}{2(1+\nu)}}) \tag{39}$$

then

$$\max_{\overline{G}_h} |u(x,t) - z(x,t)| \leq M(h^{\nu_1} + \tau^{\nu_1}), \tag{40}$$

for any $\nu_1 \in (0, 1/3)$. Thus, we arrive at the following conclusion about the ε-uniform convergence of the fitted scheme (21), (28).

Theorem 0.1 For $k + 2k_0 \leq 4$, let the estimate (14) hold for the functions $U(x,t)$ that represents the continuous component of u in (11). Then, under condition (37), the solution of the difference scheme (21), (28) converges on $\overline{G}_h$ in the discrete ℓ^∞-norm to the solution of problem (1) uniformly in ε. Under the conditions (37) or (39) respectively, the estimates (38) or (40) hold for the solution of the difference problem.

5. NUMERICAL RESULTS

By theory [1, 2] and by numerical experiments [11] it is shown that, for discontinuous initial conditions, classical difference schemes do not converge in the ℓ^∞-norm everywhere on the set $\overline{G}_h \backslash S^\star$, even for a fixed value of ε. Neither do they converge uniformly in ε in the neighbourhood of the interior layer, outside a neighbourhood of $S^\star$. However, both the true solution $u(x,t)$ and the numerical approximation $z(x,t)$ are bounded, uniformly in ε and it would be possible that the error $\max_{\overline{G}_h} |z(x,t) - u(x,t)|$ is not too large for the classical difference scheme. That would reduce the need for a special scheme.

On the other hand, the above theorem shows that the fitted scheme converges uniformly in ε on $\overline{G}_h$, but no indication is given about the value of the order constant M in (40). Moreover, the order of convergence is rather small. It might be possible that the error is rather large for any reasonable value of h or τ. This might also reduce the practical value of our fitted scheme. To decide on the practical value of the new scheme numerical experiments should give the final answer.

5.1 The model problem

To see the effect of the fitted scheme in practice, for the approximation of the model problem for singularly perturbed heat equation with a discontinuous initial condition

$$\begin{aligned} L_{(41)}u(x,t) \equiv \varepsilon^2 \frac{\partial^2}{\partial x^2} u(x,t) - \frac{\partial}{\partial t} u(x,t) = 0, \qquad & (x,t) \in G, \\ u(x,t) = \phi(x,t), \qquad & (x,t) \in S, \end{aligned} \tag{41}$$

we compare the numerical results for the classical scheme (20) and the fitted scheme (21), (28). For $v(x,t)$ in (28) for problem (41) we have

$$v(x,t) = w_0(x,t;1) + u_{(27)}(x,t),$$

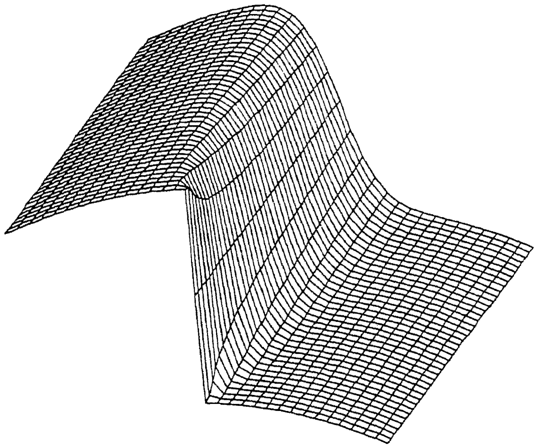

Fig.1: Computed solution with the fitted scheme.
The solution of problem (41), (43) with $u_{(41)}(x,t) = \frac{5}{2}w_0(x,t) + u_2(x,t)$; $\varepsilon = 1/8$; $N = 32$; $N_0 = 40$.

so that the coefficient $\gamma(x,t)$ in (21) take the form

$$\gamma(x,t) = \begin{cases} \dfrac{\delta_{\bar{t}} w_0(x,t) - 6\varepsilon^2 x}{\varepsilon^2 \delta_{x\bar{x}} w_0(x,t) - 6\varepsilon^2 x} & \text{for } (x,t)\in G_h, x \neq 0, \\ 1 & \text{for } (x,t)\in G_h, x = 0. \end{cases} \tag{42}$$

For $\varepsilon = 1/8$, $N = 32$, $N_0 = 40$ the solution of the model problem (41) where

$$\phi(x,t) = \frac{5}{2}w_0(x,t) + u_2(x,t),\ (x,t)\in S\,, \tag{43}$$

$$u_2(x,t) = -(x+0.5)^2 - 2\varepsilon^2 t\,, \tag{44}$$

for which we have the representation

$$u(x,t) = U(x,t) + W(x,t) = u_2(x,t) + \frac{5}{2}w_0(x,t), \quad (x,t)\in\overline{G}\backslash S^\star\,, \tag{45}$$

which is shown in Fig.1. The fitting coefficient (42) is shown in Fig.2.

We can see that the solution has a jump at $S^\star$ for $t = 0$, and for $t > 0$ it is smooth. The space derivatives of the solution are large in the neighbourhood of the interior layer. The fitted coefficient varies strongly in the neighbourhood of the set $S^\star$ and becomes almost constant (equal to 1) away from $S^\star$.

5.2 The behaviour of the numerical solution for the classical scheme

Before we see the special properties of the fitted scheme, we first show the behaviour of the classical difference scheme (20), central in x and backward in t, for the model problem (41), (43). We know that this scheme converges for a fixed parameter ε on each smooth part of the

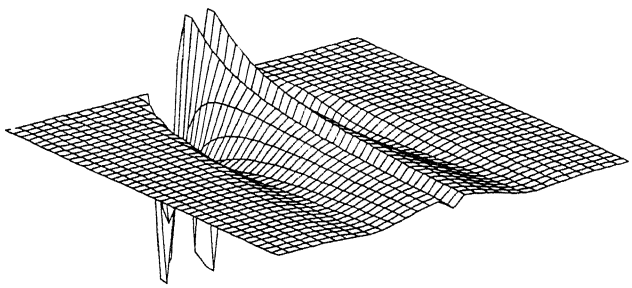

Fig.2: Coefficients $\gamma(x,t)$ in the fitted scheme.
Scheme (21), (28) for the same problem as used in Fig.1.

solution of (41), (43). Therefore we are primarily interested in the singular part of the solution to problem (41), (43) for the classical scheme. Hence, first we select the boundary conditions such that $u(x,t) = w_0(x,t)$,

$$\phi(x,t) = w_0(x,t), \quad (x,t)\in S\,. \tag{46}$$

For the approximation of problem (41), (46) we use the classical scheme (20). We solve the problem for different values of the mesh, $h = 2/N$, and the timestep, $\tau = 1/N_0$, and for different values of the small parameter ε. The results for a set numerical experiments are summarised in Table 1 and Table 2.

First we notice that –as will be obvious– asymptotically for larger N or N_0 and smaller ε, the ℓ^∞-norm of the error does not depend on ε, N and N_0 separately, but behaves as depending on a single parameter $N_0\varepsilon^{-2}$ or $N\varepsilon^{-1}$ for Table 1, and $N_0\varepsilon^{-2}$ or N for Table 2. Note that $|w_0(x,t)| \le 0.5$. From Table 1 we see that for no value of the parameter ε we can guarantee the error on $\overline{G}$ to be less than 12% for any sufficiently large N, N_0:

$$\eta_1(K,\varepsilon) = \max_{N,\ N_0\ge K}\{[\max_{(x,t)\in\overline{G}} |w_0(x,t)|]^{-1} E(N,N_0,\varepsilon)\} \ge 12\%$$

when K is sufficiently large. From the results in Table 2 we see that for no values of N_0, N we can guarantee the error on $\overline{G}$, $t \ge 0.2$ to be less than 6% for $\varepsilon\in(0,1]$:

$$\eta_2(N,N_0) = \max_{\varepsilon}\{[\max_{(x,t)\in\overline{G},t\ge 0.2} |w_0(x,t)|]^{-1} E(N,N_0,\varepsilon)\} \ge 6\%\,.$$

Thus, from the computations we observe that: (i) the classical scheme converges on the set $\overline{G}_h$ with $t \ge t_0 > 0$ for a fixed value of ε; (ii) on $\overline{G}\backslash S^\star$ the classical scheme does not converge for any fixed ε; (iii) on the set $\overline{G}_h$ with $t \ge t_0 > 0$ the scheme does not converge uniformly in ε.

5.3 The behaviour of the numerical solution for the fitted scheme

Now we study the behaviour of the fitted scheme applied to model problem (41), (43), where the function $u(x,t)$ is the sum of a smooth and a singular part

$$u(x,t) = u_2(x,t) + \frac{5}{2} w_0(x,t), \quad (x,t)\in\overline{G}\backslash S^\star\,. \tag{47}$$

Table 1. Table of errors $E(N, N_0, \varepsilon)$ for the classical scheme
In this table $E(N, N_0, \varepsilon) = \max_{(x,t)\in\overline{G}_h} |e(x, t; N, N_0, \varepsilon)|$, $e(x, t; N, N_0, \varepsilon) = z(x, t) - w_0(x, t)$ with $h = 2/N$ and $\tau = 1/N_0$.
The solution w_0 is as defined in (6) with $p_1 = 1$.

N_0		N					
		8	16	32	64	128	256
10	$\varepsilon = 1$	5.76(-2)	6.08(-2)	6.16(-2)	6.25(-2)	6.26(-2)	6.26(-2)
40		2.48(-2)	5.69(-2)	6.01(-2)	6.10(-2)	6.20(-2)	6.20(-2)
160		2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)	6.10(-2)	6.20(-2)
640		3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)	6.10(-2)
10	$\varepsilon = 1/8$	3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)	6.10(-2)
40		3.27(-2)	3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)
160		3.29(-2)	3.27(-2)	3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)
640		3.29(-2)	3.29(-2)	3.27(-2)	3.18(-2)	2.93(-2)	2.47(-2)
	ε						
$N_0 = 40$	1	2.48(-2)	5.69(-2)	6.01(-2)	6.10(-2)	6.20(-2)	6.20(-2)
	0.5	2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)	6.10(-2)	6.20(-2)
	2^{-2}	3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)	6.10(-2)
	2^{-3}	3.27(-2)	3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)
	2^{-4}	2.70(-2)	3.27(-2)	3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)
	2^{-5}	7.69(-3)	2.70(-2)	3.27(-2)	3.18(-2)	2.93(-2)	2.47(-2)
	2^{-6}	1.95(-3)	7.69(-3)	2.70(-2)	3.27(-2)	3.18(-2)	2.93(-2)
	2^{-7}	4.88(-4)	1.95(-3)	7.69(-3)	2.70(-2)	3.27(-2)	3.18(-2)
	2^{-8}	1.22(-4)	4.88(-4)	1.95(-3)	7.69(-3)	2.70(-2)	3.27(-2)
	2^{-9}	3.05(-5)	1.22(-4)	4.88(-4)	1.95(-3)	7.69(-3)	2.70(-2)

Because the problem is linear, we can study both parts of the error independently. First we consider the behaviour of the fitted scheme for the singular part, i.e for the model problem with

$$\phi(x,t) = w_0(x,t)\,, \quad (x,t)\epsilon\overline{G}\backslash S^\star\,, \tag{48}$$

as we did for the classical scheme. This initial function $w_0(x,t)$ is representative for any initial function with a discontinuity. For problem (41), (48) we have the solution

$$u(x,t) = w_0(x,t), \quad (x,t)\epsilon\overline{G}\backslash S^\star\,. \tag{49}$$

Then, considering the smooth part of the solution in the expression (47) we study problem (41) with

$$\phi(x,t) = u_2(x,t) = -(x+0.5)^2 - 2\varepsilon^2 t\,, \qquad (x,t)\epsilon\overline{G}\,. \tag{50}$$

For problem (41), (50), we have the solution

$$u(x,t) = u_2(x,t), \quad (x,t)\epsilon\overline{G}\,. \tag{51}$$

The results of the numerical experiments are given in the Tables 3 - 4.

From the results in the tables 3 - 4 we see that errors for singular and regular parts, $w_0(x,t)$ and $u_2(x,t)$, vanish with increasing $\overline{N}$, where $\overline{N} = \min(N, N_0)$ for a fixed value of the parameter $\varepsilon = 2^{-K}$, $K = 0, 1, \cdots$. Also the errors vanish with increasing $\overline{N}$ uniformly in ε. The relative error is guaranteed less than 1% for $N \geq 8$, $N_0 \geq 160$, $\varepsilon = 2^{-K}$, $K \geq 0$ when $u(x,t) = w_0(x,t)$.

Table 2. Table of errors $E_{0.2}(N, N_0, \varepsilon)$ for the classical scheme
In this table $E_{0.2}(N, N_0, \varepsilon) = \max_{(x,t)\in\overline{G}_h, t\geq 0.2} |e(x,t;N,N_0,\varepsilon)|$, $e(x,t;N,N_0,\varepsilon) = z(x,t) - w_0(x,t)$ with $h = 2/N$ and $\tau = 1/N_0$.
The solution w_0 is as defined in (6) with $p_1 = 1$.

N_0		N = 8	16	32	64	128	256
10	$\varepsilon = 1$	3.08(-2)	3.39(-2)	3.40(-2)	3.40(-2)	3.40(-2)	3.40(-2)
40		1.01(-2)	9.37(-3)	9.28(-3)	9.22(-3)	9.21(-3)	9.21(-3)
160		3.77(-3)	2.73(-3)	2.45(-3)	2.38(-3)	2.37(-3)	2.36(-3)
640		2.12(-3)	9.97(-4)	6.98(-4)	6.22(-4)	6.02(-4)	5.98(-4)
10	$\varepsilon = 1/8$	3.18(-2)	2.05(-2)	2.47(-2)	3.01(-2)	3.32(-2)	3.33(-2)
40		3.27(-2)	2.45(-2)	7.67(-3)	8.62(-3)	8.59(-3)	8.65(-3)
160		3.29(-2)	2.56(-2)	7.40(-3)	2.59(-3)	2.29(-3)	2.20(-3)
640		3.29(-2)	2.50(-2)	7.57(-3)	2.17(-3)	7.35(-4)	5.89(-4)
	ε						
$N_0 = 40$	1	1.01(-2)	9.37(-3)	9.28(-3)	9.22(-3)	9.21(-3)	9.21(-3)
	0.5	7.67(-3)	8.62(-3)	8.59(-3)	8.65(-3)	8.62(-3)	8.61(-3)
	2^{-2}	2.45(-2)	7.67(-3)	8.62(-3)	8.59(-3)	8.65(-3)	8.62(-3)
	2^{-3}	3.27(-2)	2.45(-2)	7.67(-3)	8.62(-3)	8.59(-3)	8.65(-3)
	2^{-4}	2.66(-2)	3.27(-2)	2.45(-2)	7.67(-3)	8.62(-3)	8.59(-3)
	2^{-5}	7.50(-3)	2.66(-2)	3.27(-2)	2.45(-2)	7.67(-3)	8.62(-3)
	2^{-6}	1.90(-3)	7.50(-3)	2.66(-2)	3.27(-2)	2.45(-2)	7.67(-3)
	2^{-7}	4.76(-4)	1.90(-3)	7.50(-3)	2.66(-2)	3.27(-2)	2.45(-2)
	2^{-8}	1.19(-4)	4.76(-4)	1.90(-3)	7.50(-3)	2.66(-2)	3.27(-2)
	2^{-9}	2.98(-5)	1.19(-4)	4.76(-4)	1.90(-3)	7.50(-3)	2.66(-2)

The relative error is guaranteed less than 1% for the same parameters when $u(x,t) = u_2(x,t)$.

The functions $\frac{5}{2}w_0(x,t)$ and $u_2(x,t)$ are components of the solution of the problem (41), (43). Thus we have: (i) for model problem (41), (43) the numerical scheme converges for a fixed ε in the discrete ℓ^∞-norm on $\overline{G}_h$; (ii) we observe ε-uniform convergence for the model problem (41), (43); (iii) a relative error for the model problem is guaranteed less than 2% .

5.4 The observed order of convergence

To determine the quality of the convergence, using the data from the Tables 3 and 4 we can examine the experimental order of convergence of the fitted scheme.

When we use the classical scheme (20) for problem (41), (51) then, according to the classical theory, we typically find an estimate of the form

$$\max_{\overline{G}_h\setminus S^*} |u_2(x,t) - z_{(20)}(x,t)| \leq Q(\varepsilon)(h^2 + \tau), \ (x,t)\in\overline{G}_h. \tag{52}$$

This estimate means that function $z_{(20)}(x,t)$ converges to function $u_2(x,t)$ for fixed value of ε. The constant $Q(\varepsilon)$ increases for $\varepsilon \to 0$.

From theory we know that the solution of the fitted difference scheme (21), (28) $z(x,t)$ converges ε-uniformly to the solution of problem (41), (51). To investigate the ε-uniform convergence of a function $z(x,t) = z(x,t;\varepsilon,h,\tau)$, it is natural to express an error estimate in

Table 3. Table of errors $E(N, N_0, \varepsilon)$ for the new scheme.
In this table the scheme (21) is used to solve a problem (41), (48) with an interior layer. $E(N, N_0, \varepsilon) = \max_{(x,t)\in\overline{G}_h} |e(x, t; N, N_0, \varepsilon)|$, $e(x, t; N, N_0, \varepsilon) = z(x, t) - w_0(x, t)$ with $h = 2/N$ and $\tau = 1/N_0$; the solution w_0 is as defined in (6) with $p_1 = 1$.

N_0		N 8	16	32	64	128	256
10	$\varepsilon = 1$	2.26(-2)	1.96(-2)	1.89(-2)	1.87(-2)	1.87(-2)	1.87(-2)
40		1.27(-2)	1.06(-2)	1.01(-2)	1.01(-2)	1.00(-2)	1.00(-2)
160		7.74(-3)	5.30(-3)	4.30(-3)	4.16(-3)	4.08(-3)	4.07(-3)
640		6.13(-3)	3.01(-3)	1.79(-3)	1.43(-3)	1.34(-3)	1.31(-3)
10	$\varepsilon = 1/8$	5.46(-3)	3.01(-3)	1.79(-3)	1.43(-3)	1.34(-3)	1.31(-3)
40		5.56(-3)	2.30(-3)	9.47(-4)	5.28(-4)	4.17(-4)	3.88(-4)
160		5.57(-3)	2.12(-3)	7.00(-4)	2.64(-4)	1.44(-4)	1.12(-4)
640		5.58(-3)	2.07(-3)	6.36(-4)	1.92(-4)	6.90(-5)	3.66(-5)
	ε						
$N_0 = 40$	1	1.27(-2)	1.07(-2)	1.01(-2)	1.01(-2)	1.00(-2)	1.00(-2)
	0.5	7.74(-3)	5.30(-3)	4.30(-3)	4.16(-3)	4.08(-3)	4.07(-3)
	2^{-2}	6.13(-3)	3.01(-3)	1.79(-3)	1.43(-3)	1.34(-3)	1.31(-3)
	2^{-3}	5.56(-3)	2.30(-3)	9.47(-4)	5.28(-4)	4.17(-4)	3.88(-3)
	2^{-4}	4.48(-3)	1.70(-3)	6.56(-4)	2.60(-4)	1.44(-4)	1.12(-4)
	2^{-5}	1.23(-3)	6.55(-4)	3.69(-4)	1.46(-4)	5.90(-5)	3.27(-5)
	2^{-6}	3.08(-4)	1.79(-4)	8.37(-5)	5.67(-5)	2.63(-5)	1.12(-5)
	2^{-7}	7.71(-5)	4.47(-5)	2.28(-5)	1.05(-5)	7.77(-6)	4.25(-6)
	2^{-8}	1.93(-5)	1.12(-5)	5.71(-6)	2.86(-6)	1.34(-6)	1.15(-6)
	2^{-9}	4.82(-6)	2.80(-6)	1.43(-6)	7.15(-7)	3.58(-7)	2.09(-7)

the form

$$\max_{\varepsilon} \max_{\overline{G}_h\setminus S^*} |u(x, t, \varepsilon) - z(x, t; \varepsilon, h, \tau)| \le M(h^2 + \tau)^{\nu}, \tag{53}$$

where ν does not depend on the parameters ε, h or τ. To compute ν we shall use an inequality of the form

$$\max_{\overline{G}_h\setminus S^*} |u(x, t, \varepsilon) - z(x, t; \varepsilon, h, \tau)| \le M(h^2 + \tau)^{\nu(\varepsilon)}. \tag{54}$$

We call $\nu(\varepsilon)$ in expression (54) for the error the *generalised order of convergence for a fixed value of the parameter* ε, and ν in expression (53) the *generalised order of ε-uniform convergence.*

We determine the experimental generalised order *in the point* (N, N_0) as

$$\overline{\nu}(N, N_0, \varepsilon) = (\ln E(N, N_0, \varepsilon) - \ln E(2N, 4N_0, \varepsilon))/\ln 4, \tag{55}$$

where $E(N, N_0, \varepsilon) = \max_{\overline{G}_h\setminus S^*} |u(x, t, \varepsilon) - z(x, t; \varepsilon, h, \tau)|$, $hN = 2$ and $\tau N_0 = 1$. We introduce the *experimental generalised order of convergence* for fixed ε as

$$\overline{\nu}(\varepsilon) = \min_{N,N_0} \overline{\nu}(N, N_0, \varepsilon), \tag{56}$$

Table 4. Table of errors $E(N, N_0, \varepsilon)$.
In this table the scheme (21) is used to solve a problem (41), (50) with a smooth solution. In this table $E(N, N_0, \varepsilon) = \max_{(x,t)\in\overline{G}_h} |e(x,t; N, N_0, \varepsilon)|$, $e(x,t;N,N_0,\varepsilon) = z(x,t) - u_2(x,t)$ with $h = 2/N$ and $\tau = 1/N_0$; the solution u_2 is as defined in (51).

N_0	ε	N					
		8	16	32	64	128	256
10	$\varepsilon = 1$	5.10(-2)	8.72(-2)	1.16(-1)	1.36(-1)	1.47(-1)	1.53(-1)
40		1.46(-2)	2.27(-2)	3.15(-2)	3.89(-2)	4.50(-2)	4.87(-2)
160		7.19(-3)	5.87(-3)	7.00(-3)	8.44(-3)	9.83(-3)	1.10(-2)
640		7.32(-3)	4.05(-3)	2.74(-3)	2.22(-3)	2.08(-3)	2.32(-3)
10	$\varepsilon = 1/8$	7.32(-3)	4.05(-3)	2.74(-3)	2.22(-3)	2.08(-3)	2.32(-3)
40		7.44(-3)	3.17(-3)	1.64(-3)	1.03(-3)	8.52(-4)	8.00(-4)
160		7.46(-3)	2.94(-3)	1.25(-3)	6.06(-4)	3.74(-4)	3.04(-4)
640		7.46(-3)	2.89(-3)	1.15(-3)	4.63(-4)	2.09(-4)	2.55(-4)
	ε						
$N_0 = 40$	1	1.46(-2)	2.27(-2)	3.14(-2)	3.89(-2)	4.50(-2)	4.87(-2)
	0.5	7.19(-3)	5.87(-3)	7.00(-3)	8.44(-3)	9.83(-3)	1.10(-2)
	2^{-2}	7.32(-3)	4.05(-3)	2.74(-3)	2.22(-3)	2.08(-3)	2.32(-3)
	2^{-3}	7.44(-3)	3.17(-3)	1.64(-3)	1.03(-3)	8.52(-4)	8.00(-4)
	2^{-4}	5.98(-3)	2.39(-3)	1.25(-3)	6.06(-4)	3.74(-4)	3.04(-4)
	2^{-5}	1.64(-3)	1.75(-3)	1.00(-3)	4.59(-4)	2.09(-4)	1.26(-4)
	2^{-6}	4.11(-4)	4.77(-4)	4.47(-4)	3.04(-4)	1.47(-4)	6.90(-5)
	2^{-7}	1.03(-4)	1.19(-4)	1.22(-4)	1.12(-4)	8.30(-5)	4.45(-5)
	2^{-8}	2.57(-5)	2.98(-5)	3.04(-5)	3.05(-5)	2.80(-5)	2.17(-5)
	2^{-9}	6.42(-6)	7.45(-6)	7.61(-6)	7.63(-6)	7.63(-6)	7.01(-6)

and the *experimental generalised order of ε-uniform convergence* as

$$\overline{\nu} = \min_{\varepsilon} \overline{\nu}(\varepsilon)\,. \tag{57}$$

Similarly the the *experimental ε-uniform generalised order in the point* (N, N_0) is

$$\overline{\nu}(N, N_0) = \min_{\varepsilon} \overline{\nu}(N, N_0, \varepsilon)\,. \tag{58}$$

In the Tables 6 and 5 the results are given.

From the results in the Tables 5 and 6 we see: (i) for $w_0(x,t)$ and $u_2(x,t)$ the experimental generalised order of ε-uniform convergence for the fitted scheme is approximately 0.413 and 0.450 respectively; (ii) for $N \geq 16$ and $N_0 \geq 40$ the generalised orders of ε-uniform convergence for $w_0(x,t)$ and $u_2(x,t)$ are apparently not less than 0.50. This means that in practice

$$\max_{\overline{G}_h} |u(x,t) - z(x,t)| \leq M(h + \tau^{1/2})$$

for $N \geq 16$ and $N_0 \geq 40$, $0 < \varepsilon \leq 1$, for each value of the parameter ε. Not in contradiction with the theory, for each value of ε the experimental generalised order of convergence tends to 1 for vanishing h and τ. Thus, the experimental generalised order of convergence for the fitted scheme (21), (28) for the full model problem (41), (43) is not less than predicted by the theory. The behaviour of the errors $e(x,t;N,N_0,\varepsilon) = z(x,t) - u(x,t)$ for the fitted scheme (21), (28) and for the classical scheme (20) are shown in Figs.3 and 4. We can see that the largest

Table 5. Experimental generalised order of convergence $\overline{\nu}(N, N_0, \varepsilon)$.
The fitted scheme (21), (28) for the problem (41), (48), applied to the solution $u(x,t) = w_0(x,t)$ with the interior layer. $\overline{\nu}(N, N_0, \varepsilon) = (\ln E(N, N_0, \varepsilon) - \ln E(2N, 4N_0, \varepsilon))/\ln 4$,
$E(N, N_0, \varepsilon)$ from Table 3.

N_0		N				
		8	16	32	64	128
10	$\varepsilon = 1$	0.544	0.479	0.454	0.450	0.450
40		0.631	0.653	0.640	0.650	0.651
160		0.681	0.782	0.792	0.818	0.818
10	$\varepsilon = 1/8$	0.625	0.834	0.882	0.891	0.892
40		0.696	0.858	0.922	0.938	0.945
160		0.714	0.868	0.932	0.967	0.987
	ε					
$N_0 = 40$	1	0.631	0.653	0.640	0.650	0.651
	0.5	0.681	0.782	0.792	0.818	0.818
	2^{-2}	0.708	0.834	0.882	0.891	0.892
	2^{-3}	0.854	0.904	0.934	0.939	0.946
	2^{-4}	1.387	1.104	1.084	1.070	1.067
	2^{-5}	1.391	1.484	1.351	1.237	1.195
	2^{-6}	1.392	1.485	1.498	1.433	1.314
	2^{-7}	1.393	1.485	1.498	1.486	1.378
	2^{-8}	1.393	1.485	1.498	1.500	1.340

Table 6. Experimental generalised order of convergence $\overline{\nu}(N, N_0, \varepsilon)$.
$\overline{\nu}(N, N_0, \varepsilon) = (\ln E(N, N_0, \varepsilon) - \ln E(2N, 4N_0, \varepsilon))/\ln 4$, $E(N, N_0, \varepsilon)$ from Table 4.
Computation with the new scheme (21), (28) for the smooth solution $u(x,t) = u_2(x,t)$.

N_0	ε	N				
		8	16	32	64	128
10	$\varepsilon = 1$	0.583	0.736	0.789	0.795	0.798
40		0.656	0.850	0.949	0.992	1.016
160		0.413	0.551	0.828	1.012	1.041
10	$\varepsilon = 1/8$	0.604	0.652	0.702	0.691	0.687
40		0.668	0.669	0.719	0.733	0.744
160		0.685	0.676	0.718	0.769	0.276
	ε	N				
$N_0 = 40$	1	0.656	0.850	0.949	0.992	1.016
	2^{-1}	0.413	0.551	0.828	1.012	1.041
	2^{-2}	0.603	0.652	0.702	0.691	0.687
	2^{-3}	0.820	0.669	0.719	0.733	0.744
	2^{-4}	0.887	0.627	0.724	0.769	0.783
	2^{-5}	0.891	0.984	0.859	0.820	0.799
	2^{-6}	0.892	0.985	0.998	0.938	0.863
	2^{-7}	0.893	0.985	0.998	0.999	0.966
	2^{-8}	0.893	0.985	0.998	1.000	1.000

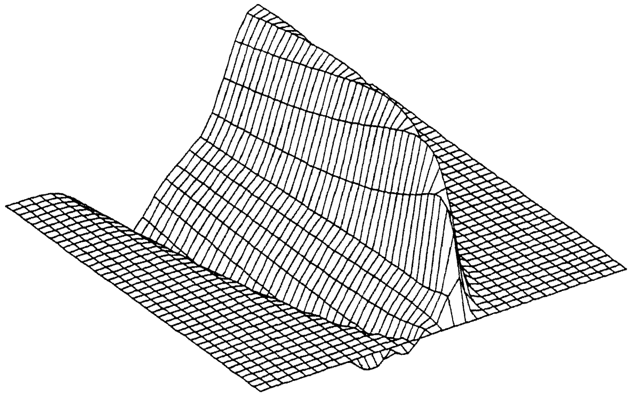

Fig.3: Discretisation error the fitted scheme.
Scheme (21), (28) is used for the same problem as used in Fig.1.

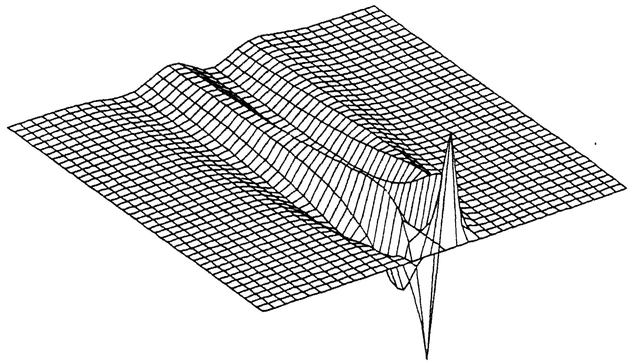

Fig.4: Discretisation error the classical scheme.
Scheme (20) is used for the same problem as used for Fig.1.

errors are in the neighbourhood of the set $S^\star$ and that the errors for the classical scheme are significantly larger than for the fitted scheme.

6. SUMMARY

For a singularly perturbed boundary value problem of parabolic type with discontinuous initial condition (1), we have constructed a specially fitted difference scheme that converges in $\overline{G}\backslash S^\star$ ε-uniformly in the ℓ^∞-norm.

Numerical experiments for a model boundary value problem with discontinuous boundary function show that a classical difference scheme does not converge ε-uniformly. Moreover, for a fixed value of ε this scheme doesn't converge in the ℓ^∞-norm in the neighbourhood of the discontinuity, and away from the discontinuity it does not converge ε-uniformly in the neighbourhood of the interior layer. In the case of the constant coefficient problem and a simple discontinuity, for which the error-function is the solution, we find that an error less than

6% on $\overline{G}$, $t \geq t_0 = 0.2$, or less than 12% on $\overline{G}\backslash S^*$ cannot be guaranteed for any small h and τ.

By theory and by numerical experiments it is also shown, that the fitted difference scheme converges ε-uniformly in the ℓ^∞-norm on $\overline{G}_h$. Moreover, for the fitted scheme, for a model problem an experimental generalised order of convergence of not less than 0.5 is observed if $h \leq 1/8$ and $\tau \leq 0.025$ e.g. $\nu(\varepsilon, N, N_0) \geq 0.5$ at $N \geq 16$, $N_0 \geq 40$. The experimental generalised order of convergence is substantially larger than the bound guaranteed by the theory. Both for the singular and for the regular part of the solution an error less than 1% is guaranteed already for $N \geq 8$, $N_0 > 40$ for any $\varepsilon \in (0, 1]$.

REFERENCES

[1] G.I.Shishkin (1988) A difference scheme for a singularly perturbed equation of parabolic type with discontinuous boundary conditions; J. Vychisl. Mat. i Mat. Fis., 28:1649–1662 (In Russian)

[2] G.I.Shishkin (1988) A difference scheme for a singularly perturbed equation of parabolic type with discontinuous initial conditions; Dokl. Akad. Nauk., 300:1066–1070 (In Russian)

[3] P.W.Hemker, G.I.Shishkin (1992) Discrete approximation of singularly perturbed parabolic pdes with a discontinuous initial condition; *Proceedings of the 6th BAIL Conference, 1992*

[4] V.B.Andreev (1980) Grid approximations of non-smooth solutions to differential equations; Differencialnie Uravnen, 26:1172–1184 (In Russian)

[5] A.V.Lykov (1978) *Heat and Mass Transfer*; Energia 1978 (In Russian)

[6] M.H.Protter, H.F.Weinberger (1967) *Maximum Principles in Differential Equations*; Prentice-Hall 1967

[7] A.Friedman (1964) *Partial Differential Equations of Parabolic Type*; Prentice Hall 1964

[8] O.A.Ladyzhenskaya, V.A.Solonnicov, N.N.Ural'tseva (1968) *Linear and Quasi-linear Equations of Parabolic type*; AMS 1968

[9] G.I.Marchuk (1982) *Methods of Numerical Mathematics.* Springer Verlag 1982

[10] A.A.Samarsky (1977) *Theory of Difference Schemes*; Nauka 1977 (In Russian)

[11] P.W.Hemker, G.I.Shishkin (1993) Approximation of parabolic pdes with a discontinuous initial condition; East-West J. Numer. Math., 1993